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Local Exponents of Two-coloured Bi-cycles whose Lengths Differ by 1

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ABSTRACT

A two-coloured digraph $D^{(2)}$ is a digraph each of whose arc is coloured by red or blue. An (h,k)-walk in a two-coloured digraph is a walk of length $(h+k)$ consisting of h red arcs and k blue arcs. A two-coloured digraph $D^{(2)}$ is primitive provided that for each pair of vertices *u* and *v* there exists an (h,k)-walk from *u* to *v*. The inner local exponent of a vertex *v* in $D^{(2)}$, denoted as $\exp\text{in}(v,D^{(2)})$, is the smallest positive integer $h+k$ over all nonnegative integers h and k such that for each vertex u in $D^{(2)}$ there is an (h,k)-walk from *u* to *v*. We study the inner local exponent of primitive two-coloured digraphs consisting of exactly two cycles of length $s+1$ and s, respectively. Let u_0 be the vertex of indegree 2 in $D^{(2)}$. For each vertex v in $D^{(2)}$, we show that $\exp\text{in}(v,D^{(2)}) = \exp\text{in}(u_0,D^{(2)}) + d(u_0,v)$ where $d(u_0,v)$ is the distance from u_0 to v.

Keywords: primitive digraph, two-coloured digraph, local exponent, bi-cycles.

1. Introduction

Let D be a digraph. A walk of length k from u to v is a sequence of arcs of the form $u \to v_1, v_1 \to v_2, \ldots, v_{k-1} \to v$. We use the notation $u \xrightarrow{k} v$ walk to represent a walk of length k from u to v. A $u \to v$ path is a $u \to v$ walk with distinct vertices except possibly $u = v$. A cycle is a $u \to v$ path with $u = v$. The distance from vertex u to vertex *v*, denoted by

$d(u,v)$, is the length of the shortest $u \to v$ path. A digraph D is strongly connected if for each pair of vertices u and v there is a $u \to v$ walk and a $v \to u$ walk. A bi-cycles is a strongly connected digraph consisting of exactly two cycles.

A strongly connected digraph D is primitive provided there is a positive integer k such that for each pair of vertices u and v there exists a $u \xrightarrow{k} v$ walk. The smallest of such positive integer k is the exponent of D and is denoted by $\exp(D)$. Brualdi and Liu, 1990, introduced the notion of local exponent of digraph. Let D be a primitive digraph on n vertices $\{v_1, v_2, \ldots, v_n\}$. The local exponent of a vertex v_t in D, denoted as $\exp(v_t, D)$, is the smallest positive integer m such that for each vertex $v_i, i = 1, 2, \ldots, n,$ in D there is a $v_t \xrightarrow{m} v_i$ walk in D.

A two-coloured digraph $D^{(2)}$ is a digraph such that each of its arcs is coloured by either red or blue. For nonnegative integers h and l with $h + l > 0,$ an (h,l)-walk in a two-coloured digraph $D^{(2)}$ is a walk consisting of h red arcs and l blue arcs. An (h,l)-walk from u to v is also denoted by $u \xrightarrow{(h,l)} v$ walk. For a walk W in $D^{(2)}$ we denote $r(W)$ to be the number of red arcs in W and $b(W)$ to be the number of blue arcs in W. The length of W is $l(W) = r(W) + b(W)$ and the vector $\begin{bmatrix} r(\mathrm{W}) \\ b(\mathrm{W}) \end{bmatrix}$ is the composition of the walk W.

The notions of primitivity and exponent of digraphs have been generalized to that of two-coloured digraphs. A strongly connected two-coloured digraph $D^{(2)}$ is primitive provided that there exist nonnegative integers h and l such that for each pair of vertices u and v in $D^{(2)}$ there is an (h,l)-walk from u to v (Fornasini and Valcher, 1998). The smallest of such positive integer $h + l$ is the *exponent* of $D^{(2)}$ and is denoted by $\exp(D^{(2)})$ (Shader and Suwilo, 2003).

For a primitive two-coloured digraph on n vertices $\{v_1, v_2, \ldots, n\}$, we define the *inner local exponent* of the vertex v_t, denoted as $\mathrm{expin}(v_t, D^{(2)})$, to be the smallest positive integer $h + l$ over all pairs of nonnegative integers h and l such that for each vertex $v_i, i = 1, 2, \ldots, n,$ there is a

$v_i \xrightarrow{(h,l)} v_t$ walk. See a similar definition of local exponent in Gao and Shao, 2009.

We discuss the inner local exponents of primitive two-coloured bi-cycles whose lengths differ by 1. In Section 2, we discuss a lower and an upper bound for local exponent and primitivity of two-coloured bi-cycles whose lengths differ by 1. In Section 3, we present main result on the inner local exponents of two-coloured bi-cycles.

2. Necessary Background

In this section we discuss primitivity of two-coloured bi-cycles whose length differ by 1. We then discuss a lower and an upper bound for the local exponents.

Let $D^{(2)}$ be a strongly connected two-coloured digraph and let $g \geq 2$ be a positive integer. Let the set of all cycles in $D^{(2)}$ be $C = \{C_1, C_2, \ldots, C_g\}$. We define a cycle matrix of $D^{(2)}$ to be a 2 by *g* matrix

$$M = \begin{bmatrix} r(C_1) & r(C_2) & \cdots & r(C_g) \\ b(C_1) & b(C_2) & \cdots & b(C_g) \end{bmatrix},$$

that is *M* is a matrix such that the ith column of *M* is the composition of the ith cycle $C_i, i = 1, 2, \ldots, g$. The content of *M* is 0 whenever the rank of *M* is 1, and the content of *M* is the greatest common divisors of the determinants of 2 by 2 submatrices of *M*, otherwise. *A* two-coloured digraph $D^{(2)}$ is primitive if and only if the content *M* is 1 (Fornasini and Valcher, 1998).

Let $D^{(2)}$ be a bi-cycles consisting of the cycle

$$C_1 : v_1 \to v_2 \to \cdots \to v_c \to v_{s+1} \to v_{s+2} \to \cdots \to v_{n=2s+1-c} \to v_1$$

of length $s+1$, and the cycle

$$C_2 : v_1 \to v_2 \to \cdots \to v_c \to v_{c+1} \to \cdots \to v_{s-1} \to v_s \to v_1$$

of length *s* for some positive integer *s*. As a direct consequence of the result of Fornasini and Valcher, 1998, the following is an algebraic characterization on the primitivity of two-coloured bi-cycles.

Corollary 2.1. *Let $D^{(2)}$ be a strongly connected primitive two-coloured bi-cycles with cycles of lengths $s+1$ and s, respectively. The cycle matrix of $D^{(2)}$ is either of the form*

$$\begin{bmatrix} s & s-1 \\ 1 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 1 \\ s & s-1 \end{bmatrix}.$$

Since changing all arcs colouring from red to blue and from blue to red does not alter the local exponents, we may assume without loss of generality that the cycle matrix of $D^{(2)}$ is the matrix

$$M = \begin{bmatrix} s & s-1 \\ 1 & 1 \end{bmatrix}. \tag{1}$$

From (1) we conclude that either $D^{(2)}$ has two blue arcs or $D^{(2)}$ has only one blue arc.

The following two propositions, due to Suwilo, 2011, will be useful in order to determine an upper bound for local exponents.

Proposition 2.2. *Let $D^{(2)}$ be a primitive two-coloured bi-cycles. Suppose v_t is a vertex that belongs to both cycles. If for some positive integers h and l, there is a path P_{v_i,v_t} from v_i to v_t such that the system*

$$M\mathbf{z} + \begin{bmatrix} r(P_{v_i,v_t}) \\ b(P_{v_i,v_t}) \end{bmatrix} = \begin{bmatrix} h \\ l \end{bmatrix}$$

has nonnegative integer solution, then there is an (h,l)-walk from v_i to v_t.

Proposition 2.3. *Let $D^{(2)}$ be a primitive two-coloured digraph. Let w be a vertex in $D^{(2)}$ with the local exponent $\text{expin}(w,D^{(2)})$. Then for each vertex u in $D^{(2)}$, $\text{expin}(u,D^{(2)}) \le \text{expin}(w,D^{(2)}) + d(w,u)$.*

The following result will be useful in determining a lower bound for inner local exponent for primitive two-coloured bi-cycles

Lemma 2.4. *Let $D^{(2)}$ be a primitive two-coloured bi-cycles consisting of two cycles C_1 and C_2 with cycle matrix* $M=\begin{bmatrix} r(C_1) & r(C_2) \\ b(C_1) & b(C_2)\end{bmatrix}$ *and* $\det(M)=1$. *Suppose* $\operatorname{expin}(v_t, D^{(2)})$ *is obtained by* (h,l)*-walks*. *Then*

$$\begin{bmatrix} h \\ l \end{bmatrix} \geq M \begin{bmatrix} b(C_2)r(P_{v_i,v_t}) - r(C_2)b(P_{v_i,v_t}) \\ r(C_1)b(P_{v_j,v_t}) - b(C_1)r(P_{v_j,v_t}) \end{bmatrix}$$

for some paths P_{v_i,v_t} and P_{v_j,v_t}.

Proof. Considering a closed walk from v_t to itself, there are nonnegative integers f_1 and f_2 such that $\begin{bmatrix} h \\ l \end{bmatrix} = M \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$. Since every walk can be decomposed into a path and some cycles, then $\begin{bmatrix} h \\ l \end{bmatrix} = \begin{bmatrix} r(P_{v_i,v_t}) \\ b(P_{v_i,v_t}) \end{bmatrix} + M\mathbf{z}$, for some path P_{v_i,v_t} from v_i to v_t and some nonnegative integer vector $\mathbf{z}$. Comparing these equations we have $\mathbf{z} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} - M^{-1} \begin{bmatrix} r(P_{v_i,v_t}) \\ b(P_{v_i,v_t}) \end{bmatrix} \geq 0.$

Hence $\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \geq M^{-1} \begin{bmatrix} r(P_{v_i,v_t}) \\ b(P_{v_i,v_t}) \end{bmatrix} = \begin{bmatrix} b(C_2)r(P_{v_i,v_t}) - r(C_2)b(P_{v_i,v_t}) \\ r(C_1)b(P_{v_i,v_t}) - b(C_1)r(P_{v_i,v_t}) \end{bmatrix}.$

Thus $f_1 \geq b(C_2)r(P_{v_i,v_t}) - r(C_2)b(P_{v_i,v_t})$ for some path P_{v_i,v_t}. Similarly, we have $f_2 \geq r(C_1)b(P_{v_j,v_t}) - b(C_1)r(P_{v_j,v_t})$ for some path P_{v_j,v_t}. Therefore,

$$\begin{bmatrix} h \\ l \end{bmatrix} \geq M \begin{bmatrix} b(C_2)r(P_{v_i,v_t}) - r(C_2)b(P_{v_i,v_t}) \\ r(C_1)b(P_{v_j,v_t}) - b(C_1)r(P_{v_j,v_t}) \end{bmatrix}$$

for some paths P_{v_i,v_t} and P_{v_j,v_t}.

From Lemma 2.4 we conclude that

$$\begin{aligned}\text{expin}(v_t, D^{(2)}) \quad \geq \quad & l(C_1)(b(C_2)r(P_{v_i,v_t}) - r(C_2)b(P_{v_i,v_t})) \\ & + l(C_2)(r(C_1)b(P_{v_j,v_t}) - b(C_1)r(P_{v_j,v_t})).\end{aligned} \tag{2}$$

for some paths P_{v_i,v_t} and P_{v_j,v_t}.

3. Results

In this section we discuss the local exponents of the class of two-coloured bi-cycles $D^{(2)}$ on n vertices $\{v_1, v_2, \ldots, v_n\}$ whose cycle lengths differ by 1. For the rest of the paper let $D^{(2)}$ be a two-coloured bi-cycles whose underlying digraph is the digraphs that consists of exactly the cycle

$$C_1 : v_1 \to v_2 \to \cdots \to v_c \to v_{s+1} \to v_{s+2} \to \cdots \to v_{n=2s+1-c} \to v_1$$

of length s+1, and the cycle

$$C_2 : v_1 \to v_2 \to \cdots \to v_c \to v_{c+1} \to \cdots \to v_{s-1} \to v_s \to v_1$$

of length s for some positive integer s. We note that C_1 and C_2 have c vertices in common. By Corollary 2.1, the bi-cycles $D^{(2)}$ has at most two blue arcs. Hence we split the discussion depending on how many blue arcs $D^{(2)}$ has and the position of the blue arcs.

We begin with the case where $D^{(2)}$ has only one blue arc. Corollary 2.1 guarantees that each cycle has one blue arc. This implies the blue arc of $D^{(2)}$ must be of the form $v_x \to v_{x+1}$ for some $1 \leq x \leq c-1$.

Theorem 3.1. *Let $D^{(2)}$ be a primitive two-coloured bi-cycles consisting cycles of lengths $s+1$ and s, respectively. If $D^{(2)}$ has a unique blue arc $v_x \to v_{x+1}$ for some $1 \leq x \leq c-1$, then $\text{expin}(v_t, D^{(2)}) = s^2 - x + d(v_1, v_t)$ for all $t = 1, 2, \ldots, n$.*

Proof. We first show that $\text{expin}(D^{(2)}, v_t) \geq s^2 - x + d(v_1, v_t)$ for all $t = 1, 2, \ldots, n$. We assume that there is a $v_{x+1} \xrightarrow{(h,l)} v_t$ walk and a $v_x \xrightarrow{(h,l)} v_t$ walk, and define $q_1 = b(C_2)r(P_{v_{x+1},v_t}) - r(C_2)b(P_{v_{x+1},v_t})$ and $q_2 = r(C_1)b(P_{v_x,v_t}) - b(C_1)r(P_{v_x,v_t})$. We consider two cases depending on the position of the vertex v_t.

***Case* 1:** *The vertex v_t lies on the $v_1 \to v_x$ path.* There are two paths P_{v_{x+1},v_t} from v_{x+1} to v_t. They are an $(s-x+d(v_1,v_t),0)$-path and an $(s-x+1+d(v_1,v_t),0)$-path. Considering the $(s-x+d(v_1,v_t),0)$-path we have $q_1 = b(C_2)r(P_{v_{x+1},v_t}) - r(C_2)b(P_{v_{x+1},v_t}) = s-x+d(v_1,v_t)$. Considering the $(s-x+1+d(v_1,v_t),0)$-path we find that $q_1 = s-x+d(v_1,v_t)+1$. So we conclude that $q_1 = s-x+d(v_1,v_t)$.

There are two paths P_{v_x,v_t} from v_x to v_t. They are an $(s-x+d(v_1,v_t),1)$-path and an $(s-x+d(v_1,v_t)+1,1)$-path. Considering the $(s-x+d(v_1,v_t),1)$-path we have

$$q_2 = r(C_1)r(P_{v_x,v_t}) - b(C_1)b(P_{v_x,v_t}) = x-d(v_1,v_t).$$

Considering the $(s-x+d(v_1,v_t)+1,1)$-path we have $q_2 = x-d(v_1,v_t)-1$. So we conclude that $q_2 = x-d(v_1,v_t)-1$.

From (2) we conclude that

$$\exp\text{in}(v_t, D^{(2)}) \geq (s+1)q_1 + sq_2 = s^2 - x + d(v_1,v_t) \tag{3}$$

for all vertices v_t that lie on the $v_1 \to v_x$ path.

***Case* 2:** *The vertex v_t lies on the $v_{x+1} \to v_n$ path or $v_{x+1} \to v_s$ path.* There is a unique path P_{v_{x+1},v_t} from v_{x+1} to v_t which is a $(d(v_{x+1},v_t),0)$-path. Using this path we have $q_1 = d(v_{x+1},v_t)$. There is a unique path P_{v_x,v_t} from v_x to v_t which is a $(d(v_{x+1},v_t),1)$-path. Using this path we have $q_2 = s-d(v_{x+1},v_t)$. From (2) we conclude that

$$\exp\text{in}(v_t, D^{(2)}) \geq (s+1)q_1 + sq_2 = s^2 - x + d(v_1,v_t) \tag{4}$$

for all vertices v_t that lie on the $v_{x+1} \to v_n$ path or $v_{x+1} \to v_s$ path.

From (3) and (4) we conclude that

$$\exp\text{in}(v_t, D^{(2)}) \geq s^2 - x + d(v_1,v_t) \tag{5}$$

for all $t = 1, 2, \ldots, n$.

We next show that $\exp\text{in}(v_t, D^{(2)}) \le s^2 - x + d(v_1, v_t)$ for all $t = 1, 2, \ldots, n.$ We firts show that $\exp\text{in}(v_1, D^{(2)}) \le s^2 - x.$ That is we show that for every vertex $v_t, t = 1, 2, \ldots, n,$ there is a $v_t \xrightarrow{(h,l)} v_1$ walk with $h = s^2 - s - x + 1$ and $l = s - 1.$ By Proposition 2.2 it suffices to show that the system

$$M\mathbf{z} + \begin{bmatrix} r(P_{v_t,v_1}) \\ b(P_{v_t,v_1}) \end{bmatrix} = \begin{bmatrix} s^2 - s - x + 1 \\ s - 1 \end{bmatrix} \tag{6}$$

has nonnegative integer solution for some path P_{v_t,v_1} from v_t to $v_1.$

The solution to the system (6) is the integer vector

$$\mathbf{z} = \begin{bmatrix} s - x + sb(P_{v_t,v_1}) - r(P_{v_t,v_1}) - b(P_{v_t,v_1}) \\ x - 1 + r(P_{v_t,v_1}) - sb(P_{v_t,v_1}) \end{bmatrix}.$$

If the vertex v_t lies on the path $v_1 \to v_x$ path, then there is a $(s - d(v_1, v_t), 1)$-path from v_t to $v_1.$ Using this path we have $z_1 = s - x + d(v_1, v_t)$ and $z_2 = x - 1 - d(v_1, v_t).$ Since $s > x$ we have $z_1 > 0$ and since $d(v_1, v_t) \le x - 1$ we have $z_2 \ge 0.$ If the vertex v_t lies on the $v_{x+1} \to v_c$ path, then there is an $(s + 1 - d(v_1, v_t), 0)$-path from v_t to $v_1.$ Using this path we have $z_1 = d(v_1, v_t) - (x + 1)$ and $z_2 = x + s - d(v_1, v_t).$ Since $d(v_1, v_t) \ge x + 1$ we have $z_1 \ge 0,$ and since $s > d(v_1, v_t)$ we have $z_2 \ge x.$ If the vertex v_t lies on the $v_{c+1} \to v_s$ path, then there is an $(s - d(v_1, v_t))$-path from v_t to $v_1.$ Using this path we have $z_1 = d(v_1, v_t) - x$ and $z_2 = x - 1 + s - d(v_1, v_t).$ Since $d(v_1, v_t) > x$ we have $z_1 > 0,$ and since $s - 1 \ge d(v_1, v_t)$ we have $z_2 \ge x.$ If the vertex v_t lies on the $v_{s+1} \to v_n$ path, then there is an $(s + 1 - d(v_1, v_t), 0)$-path from v_t to $v_1.$ Using this path we have $z_1 = d(v_1, v_t) - x - 1$ and $z_2 = x + s - d(v_1, v_t).$ Since $d(v_1, v_t) > x + 1$ we have $z_1 > 0,$ and since $s \ge d(v_1, v_t)$ we have $z_2 \ge x.$

Therefore for each vertex $v_t,$ $t = 1, 2, \ldots, n,$ there is a path P_{v_t,v_1} from v_t to v_1 such that the system (6) has a nonnegative integer solution. Proposition 2.2 guarantees that for each $v_t, t = 1, 2, \ldots, n,$ there is a $v_t \xrightarrow{(h,l)} v_1$ walk with $h = s^2 - s - x + 1$ and $l = s - 1.$ Thus

$\exp\text{in}(v_1, D^{(2)}) \le s^2 - x$ and from (5) we conclude that $\exp\text{in}(v_1, D^{(2)}) = s^2 - x.$ Now Proposition 2.3 guarantees that $\exp\text{in}(v_t, D^{(2)}) \le s^2 - x + d(v_1, v_t).$

We next discuss the case where the two-coloured bi-cycles $D^{(2)}$ contains two blue arcs. The cycle matrix M in (1) implies that each cycle of $D^{(2)}$ must contain one blue arc. Let $v_x \to v_{x+1}$ be the blue arc that lies on C_1 but not on C_2 and let $v_y \to v_{y+1}$ be the blue arc that lies on C_2 but not on C_1. We first consider the case where the two blue arcs have the same initial vertex.

Theorem 3.2. *Let $D^{(2)}$ be a primitive two-coloured bi-cycles with cycles of length $s+1$ and s, respectively. If $D^{(2)}$ has two blue arcs $v_c \to v_{s+1}$ and $v_c \to v_{c+1}$, then $\exp\text{in}(v_t, D^{(2)}) = s^2 + d(v_{s+1}, v_1) + d(v_1, v_t)$ for all $t = 1, 2, \ldots, n.$*

Proof. We note that $d(v_{c+1}, v_1) = d(v_{s+1}, v_1) - 1.$ We show that $\exp\text{in}(v_t, D^{(2)}) \ge s^2 + d(v_{s+1}, v_1) + d(v_1, v_t)$ for all $t = 1, 2, \ldots, n.$ We assume there is a $v_c \xrightarrow{(h,l)} v_t$ walk and a $v_{s+1} \xrightarrow{(h,l)} v_t$ walk and define that $q_1 = b(C_2) r(P_{v_{s+1}, v_t}) - r(C_2) b(P_{v_{s+1}, v_t})$ and $q_2 = r(C_1) b(P_{v_c, v_t}) - b(C_1) r(P_{v_c, v_t}).$ We consider three cases depending on the position of the vertex $v_t.$

***Case* 1:** *The vertex v_t lies on the $v_1 \to v_c$ path.* There is a unique path P_{v_{s+1}, v_t} from v_{s+1} to v_t which is a $(d(v_{s+1}, v_1) + d(v_1, v_t), 0)$-path. Using this path we have $q_1 = d(v_{s+1}, v_1) + d(v_1, v_t).$ There are two paths P_{v_c, v_t} from v_c to $v_t.$ They are a $(d(v_{s+1}, v_1) + d(v_1, v_t) - 1, 1)$-path and a $(d(v_{s+1}, v_1) + d(v_1, v_t), 1)$-path. From the $(d(v_{s+1}, v_1) + d(v_1, v_t) - 1, 1)$-path we have $q_2 = s + 1 - d(v_{s+1}, v_1) - d(v_1, v_t).$ From the $(d(v_{s+1}, v_1) + d(v_1, v_t), 1)$-path. we have $q_2 = s - d(v_{s+1}, v_1) - d(v_1, v_t).$ Hence we conclude that $q_2 = s - d(v_{s+1}, v_1) - d(v_1, v_t).$
From (2) we conclude that

$$\exp\text{in}(v_t, D^{(2)}) \ge (s+1) q_1 + s q_2 = s^2 + d(v_{s+1}, v_1) + d(v_1, v_t) \qquad (7)$$

for each vertex v_t that lies on $v_1 \to v_c$ path.

***Case* 2:** *The vertex v_t lies on $v_{c+1} \to v_s$ path.* There is a unique path P_{v_{s+1},v_t} from v_{s+1} to v_t which is an (s+ $d(v_{c+1},v_t)$,1)-path. Using this path we have $q_1 = d(v_{c+1},v_t)+1$. There is a unique path P_{v_c,v_t} from v_c to v_t which is a $(d(v_{c+1},v_t)$,1)-path. Using this path we find that $q_2 = s - d(v_{c+1},v_t)$. From (2) we have

$$\text{expin}(v_t, D^{(2)}) \geq (s+1)q_1 + sq_2 = s^2 + s + 1 + d(v_{c+1},v_t)\ .$$

Since $d(v_{c+1},v_t) = \mathrm{d}(\mathrm{v}_{s+1},v_1) - 1 - d(v_t,v_1) = d(v_{s+1},v_1) + d(v_1,v_t) - (s+1)$, we conclude that

$$\text{expin}(v_t, D^{(2)}) \geq s^2 + d(v_{s+1},v_1) + d(v_1,v_t) \tag{8}$$

for each vertex v_t that lies on $v_{c+1} \to v_s$ path.

***Case* 3**: *The vertex v_t lies on $v_{s+1} \to v_n$ path.* There is a unique path P_{v_{s+1},v_t} from v_{s+1} to v_t which is a $(d(v_{s+1},v_t) - \mathrm{d}(\mathrm{v}_t,v_1), 0)$-path. Using this path we find that $q_1 = d(v_{s+1},v_t) - \mathrm{d}(\mathrm{v}_t,v_1)$. There is a unique path P_{v_c,v_t} from v_c to v_t which is a $(d(v_{s+1},v_t) - \mathrm{d}(\mathrm{v}_t,v_1)$,1)-path. Using this path we find that $q_2 = s - d(v_{s+1},v_t) + d(v_t,v_1)$. By Lemma 2.4 we have

$$\begin{bmatrix} h \\ l \end{bmatrix} \geq M \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} s^2 - s + d(v_{s+1},v_t) - \mathrm{d}(\mathrm{v}_t,v_1) \\ s \end{bmatrix}.$$

We consider the existence of $v_{s+1} \to v_t$ walk. Since the path P_{v_{s+1},v_t} is a $(d(v_{s+1},v_1) - d(v_t,v_1), 0)$-path, the solution to the system

$$M\mathbf{z} + \begin{bmatrix} r(P_{v_{s+1},v_t}) \\ b(P_{v_{s+1},v_t}) \end{bmatrix} = \begin{bmatrix} s^2 - s + d(v_{s+1},v_1) - \mathrm{d}(\mathrm{v}_t,v_1) \\ s \end{bmatrix}$$

is $z_1 = 0$ and $z_2 = s$. Since v_t lies on C_1 but not on C_2, there is no $(s^2 - s + d(v_{s+1},v_1) - d(v_t,v_1), s)$-walk from v_{s+1} to v_t. We note that the shortest $v_{s+1} \to v_t$ walk that consists of at least $s^2 - s + d(v_{s+1},v_1) - d(v_t,v_1)$ red arcs ant at least s blue arcs is a $(s^2 + d(v_{s+1},v_1) - d(v_t,v_1)$,s+1)-walk.

This implies $\begin{bmatrix} h \\ l \end{bmatrix} \geq \begin{bmatrix} s^2 + d(v_{s+1}, v_t) - \mathrm{d}(\mathrm{v}_t, v_1) \\ s+1 \end{bmatrix}$. We conclude that

$$\begin{aligned} \exp\text{in}(v_t, D^{(2)}) &\geq s^2 + d(v_{s+1}, v_1) + s + 1 - d(v_t, v_1) \\ &= s^2 + d(v_{s+1}, v_1) + d(v_1, v_t) \end{aligned} \tag{9}$$

for each vertex v_t that lies on $v_{s+1} \to v_n$ path.

Now from (7), (8), and (9) we conclude that

$$\exp\text{in}(v_t, D^{(2)}) \geq s^2 + d(v_{s+1}, v_1) + d(v_1, v_t) \tag{10}$$

for all $t = 1, 2, \ldots, n.$

We next show $\exp\text{in}(v_t, D^{(2)}) \leq s^2 + d(v_{s+1}, v_1) + d(v_1, v_t)$ for all $t = 1, 2 \ldots, n.$ We first show that $\exp\text{in}(v_1, D^{(2)}) \leq s^2 + d(v_{s+1}, v_1).$ That is we show that for every vertex v_t, $t = 1, 2, \ldots, n,$ there is a $(s^2 - s + d(v_{s+1}, v_1), s)$-walk from v_t to v_1. By Proposition 2.2, it suffices to show that the system

$$M\mathbf{z} + \begin{bmatrix} r(P_{v_t, v_1}) \\ b(P_{v_t, v_t}) \end{bmatrix} = \begin{bmatrix} s^2 - s + d(v_{s+1}, v_1) \\ s \end{bmatrix} \tag{11}$$

has a nonnegative integer solution for some path P_{v_t, v_1} from v_t, $t = 1, 2, \ldots n,$ to v_1.

The solution to the system (11) is the integer vector

$$\mathrm{z} = \begin{bmatrix} d(v_{s+1}, v_1) + sb(P_{v_t, v_1}) - r(P_{v_t, v_1}) - b(P_{v_t, v_1}) \\ s - d(v_{s+1}, v_1) + r(P_{v_t, v_1}) - sb(P_{v_t, v_1}) \end{bmatrix}.$$

If the vertex v_t lies on the $v_1 \to v_c$ path, then using the $(s - d(v_1, v_t), 1)$-path from v_t to v_1 we can show that $z_1 \geq 0$ and $z_2 \geq 0$. If the vertex v_t lies on the $v_{c+1} \to v_s$ path or on the $v_{s+1} \to v_n$ path, then using the $(d(v_t, v_1), 0)$-path from v_t to v_1 we can show that $z_1 \geq 0$ and $z_2 \geq 1$.

Therefore, for each vertex v_t, $t = 1, 2, \dots, n$, there is a path P_{v_t, v_1} from v_t to v_1 such that the system (11) has a nonnegative integer solution. Proposition 2.2 guarantees that for each v_t, $t = 1, 2, \dots, n$, there is a $v_t \xrightarrow{(h,l)} v_1$ walk with $h = s^2 - s + d(v_{s+1}, v_1)$ and $l = s$. This implies $\text{expin}(v_1, D^{(2)}) \le s^2 + d(v_{s+1}, v_1)$. Considering (10) we conclude that $\text{expin}(v_1, D^{(2)}) = s^2 + d(v_{s+1}, v_1)$. Proposition 2.3 implies that $\text{expin}(v_1, D^{(2)}) \le s^2 + d(v_{s+1}, v_1) + d(v_1, v_t)$ for all $t = 1, 2, \dots, n$.

We now consider the case where $D^{(2)}$ has two blue arcs with different initial vertices.

Theorem 3.3. *Let* $D^{(2)}$ *be a primitive two-coloured bi-cycles with cycles of lengths* $s+1$ *and* s*, respectively. If* $D^{(2)}$ *has two blue arcs* $v_x \to v_{x+1}$ *and* $v_y \to v_{y+1}$, *for some* $s+1 \le x \le n-1$, and $c+1 \le y \le s-1$, *then*

$$\text{expin}(v_t, D^{(2)}) = s^2 + \left| d(v_{x+1}, v_1) - d(v_{y+1}, v_1) \right| s + \max\{d(v_{x+1}, v_1), d(v_{y+1}, v_1)\} + d(v_1, v_t)$$

for all $t = 1, 2, \dots, n$.

Sketch of Proof. For simplicity we define $d_1 = d(v_{x+1}, v_1)$ and $d_2 = d(v_{y+2}, v_1)$. We consider two cases, when $d_1 > d_2$ and $d_1 \le d_2$.

***Case* 1:** $d_1 > d_2$. Similar argument to the proof of Theorem 3.1 and Theorem 3.2 can be used to show that $\text{expin}(v_t, D^{(2)}) = s^2 + (d_1 - d_2)s + d_1 + d(v_1, v_t)$. The lower bound can be found by setting up $q_1 = b(C_2)r(P_{v_{x+1}, v_t}) - r(C_2)b(P_{v_{x+1}, v_t})$ and $q_2 = r(C_1)b(P_{v_y, v_t}) - b(C_1)r(P_{v_y, v_t})$. The upper bound can be found by showing that for each vertex $v_t, t = 1, 2, \dots, n,$ the system

$$M\mathbf{z} + \begin{bmatrix} r(P_{v_t, v_1}) \\ b(P_{v_t, v_1}) \end{bmatrix} = \begin{bmatrix} s^2 + (d_1 - d_2 - 1)s + d_2 \\ s + d_1 - d_2 \end{bmatrix}$$

has a nonnegative integer solution for some path P_{v_t,v_1} from v_t to v_1. This will imply that $\exp\text{in}(v_1, D^{(2)}) = s^2 + (d_1 - d_2)s + d_1$. Proposition 2.3 implies that $\exp\text{in}(v_t, D^{(2)}) \le s^2 + (d_1 - d_2)s + d_1 + d(v_1, v_t)$ for all $t = 1, 2, \ldots, n$.

***Case* 2:** $d_1 \le d_2$. Suppose there is $v_x \xrightarrow{(h,l)} v_t$ walk and $v_{y+1} \xrightarrow{(h,l)} v_t$ walk for some vertex v_t in $D^{(2)}$. We shall show that $\exp\text{in}(v_t, D^{(2)}) = s^2 + (d_2 - d_1)s + d_2 + d(v_1, v_t)$. As in the proof of Theorem 3.1 and Theorem 3.2 the upper bound can be found by setting up $q_1 = b(C_2)r(P_{v_{y+1},v_t}) - r(C_2)b(P_{v_{y+1},v_t})$ and $q_2 = r(C_1)b(P_{v_x,v_t}) - b(C_1)r(P_{v_x,v_t})$. The lower bound can be found by first show that the system

$$Mz + \begin{bmatrix} r(P_{v_t,v_1}) \\ b(P_{v_t,v_1}) \end{bmatrix} = \begin{bmatrix} s^2 + (d_2 - d_1 - 1)s + d_1 \\ s + d_2 - d_1 \end{bmatrix}$$

has a nonnegative integer solution for some path P_{v_t,v_1} from v_t to v_1. This will imply that $\exp\text{in}(v_1, D^{(2)}) = s^2 + (d_2 - d_1)s + d_2$. Proposition 2.3 implies that $\exp\text{in}(v_t, D^{(2)}) \le s^2 + (d_2 - d_1)s + d_2 + d(v_1, v_t)$ for all $t = 1, 2, \ldots, n$.

Finally, from Case 1 and Case 2 we conclude that

$$\exp\text{in}(v_t, D^{(2)}) = s^2 + |d_1 - d_2|s + \max\{d_1, d_2\} + d(v_1, v_t)$$

for all $t = 1, 2, \ldots, n$.

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